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LETTER TO THE EDITOR

Integral representation for the eigenstates of the spin system with inverse square interactions

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Abstract. We consider the Calogero–Sutherland–Moser system with spin $-\frac{1}{2}$. We give the integral representation for the ground state wavefunction. Excited states of the Calogero spin model confined in the harmonic potential are also given.

The one-dimensional quantum many-body system with inverse square interactions has been extensively studied. Such a system is called the Calogero–Sutherland–Moser (CSM) system [1–3], whose Hamiltonian is

$$\mathcal{H} = -\sum_{j}^{N} \frac{\partial^2}{\partial x_j^2} + 2g \sum_{1\leqslant j<k\leqslant N} V(x_j - x_k)\,. \tag{1}$$

We call such systems the (i) Calogero model, $V(x) = x^{-2}$, and (ii) Sutherland model, $V(x) = \sinh^{-2} x$. One of the interesting properties of this system is that the eigenstates are the Jastrow-type wavefunction and that the particles have fractional statistics [4].

Since the independent works of Haldane [5] and Shastry [6] the spin system with inverse square exchange, which is the Haldane–Shastry model, has attracted much attention. The Hamiltonian is given by

$$\mathcal{H}_{\mathrm{HS}} = \sum_{j<k} \frac{P_{jk}}{\sin^2(\pi/L)(x_j - x_k)}\,. \tag{2}$$

This model can be viewed as the generalization of the usual Heisenberg spin chain, but its structure still remains unclear. In this letter we consider spin generalization of the Calogero–Sutherland–Moser model [7–11],

$$\mathcal{H}_{\mathrm{C}} = -\sum_{j}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{1\leqslant j<k\leqslant N} 2\frac{\kappa^2 - \kappa P_{jk}}{(x_j - x_k)^2} \tag{3}$$

$$\mathcal{H}_{\mathrm{S}} = -\sum_{j}^{N} \frac{\partial^2}{\partial z_j^2} + \sum_{1\leqslant j<k\leqslant N} 2\frac{\kappa^2 - \kappa P_{jk}}{\sinh^2(z_j - z_k)} \tag{4}$$

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and give the integral representation for the ground state wavefunction. Systems (3) and (4) are called the Calogero and the Sutherland spin models, respectively. We also consider the Calogero spin system confined in the harmonic potential,

$$\mathcal{H}_{\mathrm{CM}} = \mathcal{H}_{\mathrm{C}} + \sum_{j}^{N} \omega^2 x_j^2 . \tag{5}$$

In these Hamiltonians P_{jk} denotes the permutation operator in spin space of the jth and kth particles, $P|\alpha\rangle \otimes |\beta\rangle = |\beta\rangle \otimes |\alpha\rangle$. Generally the $su(\nu)$ spin case is integrable, but we only consider the $su(2)$ spin-$\frac{1}{2}$ case for brevity. In this case the permutation operator P can be written as

$$\begin{aligned} P &= \tfrac{1}{2}(\sigma \otimes \sigma + 1) \\ &= \sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+ + \tfrac{1}{2}\sigma^z \otimes \sigma^z + \tfrac{1}{2} . \end{aligned} \tag{6}$$

Here we have used the Pauli spin matrices σ^α,

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the creation–annihilation operators

$$\sigma^+ = \tfrac{1}{2}(\sigma^x + \mathrm{i}\sigma^y) \qquad \sigma^- = \tfrac{1}{2}(\sigma^x - \mathrm{i}\sigma^y) .$$

Recent studies have shown that the system with inverse square interactions is closely related with the classical Yang–Baxter equation. The classical Yang–Baxter equation is the functional equation,

$$\left[X^{12}(u), X^{13}(u+v)\right] + \left[X^{12}(u), X^{23}(v)\right] + \left[X^{13}(u+v), X^{23}(v)\right] = 0 \tag{7}$$

where X^{ij} signifies the matrix on $V \otimes V \otimes V$, acting as X on the ith and jth spaces. Parameters u and v are the spectral parameters. It is known that the solutions of the classical Yang–Baxter equation (7) are classified into three cases: rational, trigonometric and elliptic solutions [11]. Two-dimensional representations of the solutions are the following:

(i) rational

$$X(u) = \frac{P}{u}$$

(ii) trigonometric

$$X(u) = \coth(u)\, P + r .$$

Here we have introduced the operator r as

$$r = \sigma^+ \otimes \sigma^- - \sigma^- \otimes \sigma^+ .$$

Corresponding to the solution of the classical Yang–Baxter equation (7), we can introduce several quantum 'integrable' systems. First example is the Gaudin magnet [12–16], which is the classical limit of the inhomogeneous spin chain. The Hamiltonian has the form

$$\mathcal{H}_j = \sum_{k \neq j}^{N} X^{jk}(x_j - x_k) \tag{8}$$

where the function $X(x)$ is the solution of the classical Yang–Baxter equation (7). The integrability of this model is supported by the consistency condition

$$[\mathcal{H}_j, \mathcal{H}_k] = 0 \qquad j, k = 1, 2, \ldots, N . \tag{9}$$

This model is exactly solved in terms of the Bethe ansatz method.

The other integrable system is the following linear differential equations:

$$\nabla_j \Psi(z) = 0 \qquad j = 1, 2, \ldots, N \tag{10}$$

where the differential operators ∇_j are defined by

$$\nabla_j = \frac{\partial}{\partial z_j} - \kappa \sum_{k \neq j} X^{jk}(z_j - z_k)\,. \tag{11}$$

The parameter κ is an arbitrary constant. The integrability condition of (10),

$$[\nabla_j, \nabla_k] = 0 \qquad j, k = 1, \ldots, N \tag{12}$$

is satisfied if $X^{jk}(u)$ is an odd function of u, and satisfies the classical Yang–Baxter equation (7). From now on we shall call the system of the differential equations (10) the generalized Knizhnik–Zamolodchikov (KZ) equation. For the rational solution $X(u) = P/u$, we have the 'original' KZ equation [17] for the N-points correlation functions of the Wess–Zumino–Witten model. Particulary, we call (10) with $X(u) = \coth(u)\,P + r$ as the trigonometric KZ equation.

The relation of the KZ equation and the CSM model (1) is revealed in [7, 18, 19]. The solutions of the CSM model is given by a 'Cherednik–Matsuo' map. For the case of the CSM spin system, the eigenstates can be derived from the KZ equation as follows. By use of the trigonometric KZ operator

$$\nabla_j = \frac{\partial}{\partial z_j} - \kappa \sum_{k \neq j} (\coth(z_j - z_k) P_{jk} + r_{jk}) \tag{13}$$

the Sutherland-type spin system (4) is constructed as

$$\mathcal{H}_{\mathrm{S}} = \sum_j \nabla_j^\dagger \nabla_j - \kappa^2 \bigg(\tfrac{1}{4} N \Big(\sum_j \sigma_j^z \Big)^2 + \tfrac{1}{12} N(N^2 - 4) \bigg)\,. \tag{14}$$

Then one knows that the solution of the trigonometric KZ equation (10) is also the eigenstate of the Sutherland-type spin system.

In the same way the Calogero spin system confined in the harmonic potential (5) can be obtained by use of the following KZ operator:

$$\tilde{\nabla}_j = \frac{\partial}{\partial x_j} - \kappa \sum_{k \neq j} \frac{P_{jk}}{x_j - x_k} + \omega x_j\,. \tag{15}$$

The Hamiltonian (5) is written as

$$\mathcal{H}_{\mathrm{CM}} = \sum_j \tilde{\nabla}_j^\dagger \tilde{\nabla}_j + \omega N + \frac{\kappa \omega}{2} \bigg(N(N - 4) + \Big(\sum_j \sigma_j \Big)^2 \bigg). \tag{16}$$

Note that the the energy depends on the total spin $\boldsymbol{J} = \sum_j \boldsymbol{\sigma}_j$ while the Sutherland-type system (14) depends only on its z-component $J^z = \sum_j \sigma_j^z$.

Now to obtain the eigenstates of the system let us study the solution of the KZ equation. In general the solutions can be written in terms of the Aomoto–Gelfand–Selberg-type hypergeometric integral [20–25]. The same integral representations can be given by the method of the Wakimoto construction [26], and the 'off-shell Bethe ansatz' method [27–29]. In the method of the off-shell Bethe ansatz the solutions of the KZ equation are constructed from the quantum inhomogeneous spin chain. This fact enlightens the relationship between the Gaudin magnet (8) and the KZ equation (10). Note that the Dunkl operator is also constructed from the inhomogeneous transfer matrix [30, 31].

We only write the solution of the KZ equation, $\tilde{\nabla}_j \Phi(x) = 0$. The integral solution is given by the following integral representation:

$$\Phi(x) = \int_\gamma \mathrm{d}t\, \chi_{\mathrm{R}}(x,t)\phi(x,t) \exp\Big(-\tfrac{1}{2}\sum_j \omega x_j^2\Big) \tag{17}$$

where γ means the closed contour on the Riemann surface, and the functions χ_{R} and ϕ are given by

$$\begin{aligned}
\phi(x,t) &= \prod_\alpha^n \left(\sum_j^N \frac{1}{t_\alpha - x_j}\sigma_j^-\right)|0\rangle \\
\chi_{\mathrm{R}}(x,t) &= \prod_{j<k}^N (x_j - x_k)^\kappa \prod_{\alpha<\beta}^n (t_\alpha - t_\beta)^{2\kappa} \prod_\alpha^n \prod_j^N (t_\alpha - x_j)^{-\kappa}.
\end{aligned}$$

The state $|0\rangle$ is the fully polarized spin state

$$|0\rangle = |\uparrow\rangle_1 \otimes \cdots \otimes |\uparrow\rangle_N$$

and the total spin of $\Phi(x)$ is $J^z = \frac{1}{2}N - n$. The integration variable t_α can be considered as the spectral parameter of the quasi-particle (spinon).

The solution of the trigonometric KZ equation, $\nabla_j \Psi(z) = 0$, can be given by the integral form

$$\Psi(z) = \int_\gamma \mathrm{d}\lambda\, \chi_{\mathrm{T}}(\lambda, z)\psi(\lambda, z) \tag{18}$$

where

$$\begin{aligned}
\psi(z,\lambda) &= \prod_\alpha^n \left(\sum_k^N (1 + \coth(\lambda_\alpha - z_k))\sigma_k^-\right)|0\rangle \\
\chi_{\mathrm{T}}(z,\lambda) &= \prod_{j<k}^N (\sinh(z_j - z_k))^\kappa \prod_{\alpha<\beta}^n (\sinh(\lambda_\alpha - \lambda_\beta))^{2\kappa} \prod_\alpha^n \prod_j^N (\sinh(\lambda_\alpha - z_j))^{-\kappa}.
\end{aligned}$$

The total spin of the state $\Psi(z)$ is $J^z = \frac{1}{2}N - n$.

For the ferromagnetic case ($n = 0$), the model is the same as the spinless case. In this case wavefunctions $\Phi(x)$ and $\Psi(z)$ reduce to the simple Jastrow-type functions, $\prod_{j<k}(x_j - x_k)^\kappa$ and $\prod_{j<k}(\sinh(z_j - z_k))^\kappa$, respectively. Both of them are known as the ground state wavefunctions for the original CSM model (1). One can also easily see the fractional statistics.

The excited states for the Calogero model confined in the harmonic potential (5) can be obtained from $\Phi(x)$ [32]. We can write explicitly the creation operators C_n, which satisfy the commutation relation

$$[\mathcal{H}_{\mathrm{CM}}, C_n^\alpha] = 2n\omega C_n^\alpha. \tag{19}$$

Such creation operators are defined by

$$C_n^\alpha = \sum_{j,k} \left(\mathbf{S}^\alpha(\mathbf{L} + \omega\mathbf{X})^n\right)_{jk} \qquad \alpha = 0, x, y, z \tag{20}$$

where the operator valued $N \times N$ matrices $\mathbf{L}$, $\mathbf{X}$ and $\mathbf{S}^-$ are

$$\begin{aligned}
\mathbf{L}_{jk} &= -\mathrm{i}\frac{\partial}{\partial x_j}\delta_{jk} + \frac{\mathrm{i}\kappa P_{jk}}{x_j - x_k}(1 - \delta_{jk}) \\
\mathbf{X} &= \mathrm{diag}(\mathrm{i}x_1, \mathrm{i}x_2, \ldots, \mathrm{i}x_N) \\
\mathbf{S}^\alpha &= \mathrm{diag}(\sigma_1^\alpha, \ldots, \sigma_N^\alpha) \qquad \alpha = x, y, z
\end{aligned}$$

and $\mathbf{S}^0 \equiv 1$. Note that the commutation relation (19) is a subalgebra of the $\mathcal{W}_\infty$ algebra, and that the Hamiltonian $\mathcal{H}_{\mathrm{CM}}$ and the creation operator C_n are identified with $\mathcal{W}_0^{(2)}$ and $\mathcal{W}_n^{(1)}$, respectively. The wavefunctions $C_{n_1}^{\alpha_1} C_{n_2}^{\alpha_2} \ldots \Phi$ are also the eigenstates of (5).

Finally we comment on the symmetry of the system with inverse square interactions. The Calogero–Sutherland–Moser spin system has interesting properties, one of which is the Yangian symmetry [30]; the transfer matrix of this system has the rational R-matrix structure. In [33] it has been shown that the Wess–Zumino–Witten conformal field theory provides a natural realization of the Yangian symmetry. In fact the $\mathcal{W}_{1+\infty}$ symmetry is given for the CSM spin system [34]; the Calogero-type and the Sutherland-type models are unified as the $\mathcal{W}$ operators. In the $\mathcal{W}_{1+\infty}$ picture the Sutherland-type system (4) can be derived from the Calogero system (3). While the Calogero-type is $\mathcal{W}_2^{(1)}$, the Sutherland-type is written as $\mathcal{W}_0^{(3)}$. These $\mathcal{W}$ operators are defined recursively

$$\mathcal{W}_n^{(s)} = \frac{1}{2(n+s)} \Big[\sum_j x_j^2, \mathcal{W}_{n+2}^{(s-1)} \Big]. \tag{21}$$

This relation is quite intriguing, but we believe that this fact originates from the similarity between the rational and the trigonometric KZ equation [35]. By setting $x_j = \exp(2z_j)$ in (10) with (), we obtain the rational KZ equation,

$$\frac{\partial}{\partial x_j} \Psi = \kappa \sum_{k \neq j} \frac{P_{jk}}{x_j - x_k} \Psi + \frac{\kappa}{2x_j} h_j \Psi \tag{22}$$

where $h_j = \sum_{k \neq j} (-P_{jk} + r_{jk})$.

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